Phase diagram of planar $U(1) \times U(1)$ superconductor
Condensation of vortices with fractional flux and a superfluid state

Egor Babaev

Institute for Theoretical Physics, Uppsala University Box 803, S-75108 Uppsala, Sweden
Cornell University 620 Clark Hall Ithaca, NY 14853-2501
Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway
Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland

We discuss a phase diagram of two-dimentional $U(1) \times U(1)$ superconductor in the field theoretic formalizm of Ref. [17]. In particular we discuss that when penetration length is short the system exhibit a quasi-neutral quasi-superfluid state which is a state when quasi-long range order sets in only in phase difference while individually the phases are disordered.

INTRODUCTION

Berezinskii-Kosterlitz-Thouless (BKT) transitions and phase diagrams of planar superfluids and superconductors is a subject of wide interest and intensive studies [1]-[10]. The present study is motivated by condensed matter systems with multiple coexistent condensates [11]-[14]. In two-band superconductors a Fermi surface consists of several disconnected parts which gives rise to several types of carriers (which are "living" on different parts of the Fermi surface) and correspondingly to multiple gaps [11]-[13]. In known two-band superconductors the two types of carriers are not independently conserved and the $U(1) \times U(1)$ symmetry is softly or strongly broken to $U(1)$. The two-gap model with the exact $U(1) \times U(1)$ symmetry has been discussed in the theoretical studies of liquid metallic hydrogen [14]. In such a system (which appears being close to realization in high pressure experiments) there should be superconductivity of electron-electron and proton-proton Cooper pairs which gives rise to the $U(1) \times U(1)$ symmetry. Besides that the two-gap superconductivity is indeed a question of an abstract academic interest due to this system albeit being formally very simple has deep physics and exhibit numerous interesting counter-intuitive phenomena (see also remark [15]). We also mention that certain formal aspects of a BKT transition in a square-lattice planar two-gap Abelian Higgs model were earlier also a subject of study in context of the lattice theory which was considered in connection with spin-charge separated superconductivity [10]. However, physically, the phase diagram of two-gap system is principally different from the spin-charge separated superconductor. In the papers by Rodriguez [10] it was analyzed a two-component model which was suggested to be relevant for spin-charge separated superconductor in a presence of a coupling to statistical gauge field. The main principal distinction of the system [10] and a two-gap superconductor is that in a spin-charge separated superconductor there exist two vector potentials which correspond to statistical and ordinary gauge fields [16], [10].

TWO-GAP GINZBURG-LANDAU MODEL AND ITS REPRESENTATION IN GAUGE-INVARIANT VARIABLES

We begin with a brief outlining of basic properties of the Ginzburg-Landau functional for a two-gap (distinguished by index $\alpha = 1, 2$) planar superconductor:

$$
F = \int d^2x \left[\frac{1}{2m_1} |(\nabla + ie\mathbf{A})\Psi_1|^2 + \frac{1}{2m_2} |(\nabla + ie\mathbf{A})\Psi_2|^2 \right.
$$
$$
\left. + \eta[\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1] + V(|\Psi_{1,2}|^2) + \frac{\mathbf{B}^2}{2}, \right] \tag{1}
$$

where $\Psi_\alpha = |\Psi_\alpha|e^{i\phi_\alpha}$ and $V(|\Psi_\alpha|^2) = -b_\alpha|\Psi_\alpha|^2 + \frac{c_\alpha}{2}|\Psi_\alpha|^4$ and η is a characteristic of interband Josephson coupling strength [12]. In a general case (1) may include other potential terms and mixed gradients terms which we drop in order have discussion in the simplest form. Similar models are also discussed in particle physics [19]. The situation is quite complex in a charged system (1) since the condensates are coupled by the field $\mathbf{A}$. However the variables in (1) can actually be separated in the London limit in a simply-connected space. In [13] it was shown that the model (1) is exactly equivalent to an extended version of Faddeev $O(3)$ nonlinear σ-model [20] (which is also relevant in high energy physics [21] and for triplet superconductors [22]). The version of this model discussed in [13] consists of a three-component unit vector $\vec{\mathbf{n}}$ in interaction with a vector field $\vec{\mathsf{C}}$ and a density-related variable ρ:

$$
F = \frac{\rho^2}{4}(\nabla\vec{\mathbf{n}})^2 + (\nabla\rho)^2 + \frac{\rho^2}{16}\vec{\mathsf{C}}^2 + \rho^2 K n_1 + V(\rho, n_3)
$$
$$
+ \frac{1}{32}\left(\nabla_i \mathsf{C}_j - \nabla_j \mathsf{C}_i - \vec{\mathbf{n}} \cdot \nabla_i\vec{\mathbf{n}} \times \nabla_j\vec{\mathbf{n}}\right)^2 \tag{2}
$$

where $\nabla_i = \frac{d}{dx_i}$ and

$$
V = A + Bn_3 + Cn_3^2 \tag{3}
$$
$$
A = \rho^2[4c_1m_1^2 + 4c_2m_2^2 - b_1m_1 - b_2m_2]
$$
$$
B = \rho^2[8c_2m_2^2 - 8c_1m_1^2 - b_2m_2 + b_1m_1]
$$
$$
C = 4\rho^2[c_1m_1^2 + c_2m_2^2].
$$

A position of the unit vector $\vec{\mathbf{n}}$ on the sphere S^2 can be characterized by two angles as follows:

$$\vec{\mathbf{n}} = (\sin(\theta)\cos(\gamma_n), \sin(\theta)\sin(\gamma_n), \cos(\theta)). \quad (4)$$

The variables of (1) and (2) are related in the following way [13]:

$$\gamma_n = (\phi_1 - \phi_2); \quad (5)$$

$$|\Psi_{1,2}| = \left[\rho\sqrt{2m_1}\,\sin\left(\frac{\theta}{2}\right), \rho\sqrt{2m_2}\,\cos\left(\frac{\theta}{2}\right)\right]; \quad (6)$$

$$\vec{\mathsf{C}} = \frac{i}{m_1\rho^2}\left\{\Psi_1^*\nabla\Psi_1 - \Psi_1\nabla\Psi_1^*\right\} + \frac{i}{m_2\rho^2}\{\Psi_2^*\nabla\Psi_2$$
$$-\Psi_2\nabla\Psi_2^*\} - \frac{2e}{\rho^2}\left(\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}\right)\mathbf{A};$$

$\rho^2 K n_1 = \eta[\Psi_1^*\Psi_2 + \Psi_2^*\Psi_1]$, where $K \equiv 2\eta\sqrt{m_1 m_2}$ is the Josephson term. The field $\vec{\mathsf{C}}$ is directly related to supercurrent

$$\vec{\mathsf{C}} = \frac{\mathbf{J}}{e\rho^2} \quad (7)$$

The mass of the field $\vec{\mathsf{C}}$ is the manifestation of the Meissner effect [13], the corresponding magnetic field penetration length is:

$$\lambda^2 = \frac{1}{e^2}\left[\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}\right]^{-1} = (2e^2\rho^2)^{-1}. \quad (8)$$

The potential term V in (2) breaks exact $O(3)$ symmetry associated with $\vec{\mathbf{n}}$ to $O(2)$ and defines the ground state for n_3 (we denote it by $\tilde{n}_3$), which corresponds to uniform density of condensates [13]:

$$\tilde{n}_3 = \cos\tilde{\theta} = \left[\frac{N_2}{m_2} - \frac{N_1}{m_1}\right]\left[\frac{N_1}{m_1} + \frac{N_2}{m_2}\right]^{-1}, \quad (9)$$

where $N_{1,2}$ stands the for average concentrations of Cooper pairs $< |\Psi_{1,2}|^2 >$. The term $\rho^2 K n_1$ breaks the remaining $O(2)$ symmetry. Thus the ground state of $\vec{\mathbf{n}}$ is a point where $\cos\gamma_n = -1$ if $\eta > 0$ or $\cos\gamma_n = 1$ if $\eta < 0$ on a circle specified by the condition $n_3 = \tilde{n}_3$. In case when $\eta \approx 0$ the system has an approximate $U(1)$ symmetry associated with γ_n. With this we conclude the brief reminder of the basic properties of the model (1), more details can be found in [13, 17].

Indeed we can study the phase diagram of a two-gap superconductor either using the variables $\vec{\mathbf{n}}$ and $\vec{\mathsf{C}}$ or the initial variables $|\Psi_\alpha|e^{-i\phi_\alpha}$. The former representation is more compact and convenient, while the later variables are more traditional so below we discuss the phase diagram of the model in terms of both representations. Additional motivation for it is that besides superconductivity the model (2) is of independent interest being relevant in particle physics [21].

VORTICES IN TWO-GAP SUPERCONDUCTOR

In order to understand BKT transitions in a two-gap system one should understand nature of topological excitations. The two-gap system allows vortices with fractional flux [17] (see also the remark [18]). In this section we discuss the physical origin of the vortices in the models (1) and (2) (somewhat extending the discussion in [17]). Before we discuss vortices with fractional flux, we would like to remind briefly the origin of flux quantization in ordinary superconductors. First let us however consider a topological defect in a *neutral* $U(1)$ system like ^{4}He which is described by a complex scalar field $|\Psi|e^{i\phi}$ with free energy

$$F_{neutr} = \frac{1}{2m}|\nabla\Psi|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 \quad (10)$$

Such a system enjoys topological excitations of the $S^1 \to S^1$ map (when the phase ϕ changes $2\pi n$ around the core), in the form of vortices which have logarithmically divergent energy per unit length [23].

Let us now consider a superconductor. That is a charged system, so we have a coupling to the vector potential $\mathbf{A}$

$$F = \frac{1}{2m}|(\nabla + ie\mathbf{A})\Psi|^2 + a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{\mathbf{B}^2}{2} \quad (11)$$

Let us consider a vortex where the phase changes 2π around the core ($2\pi n$ winding of the phase is a topological requirement which needed for singlevaluedness of the order parameter). Then in contrast to neutral case, away from the vortex core we can minimize energy by compensating gradients of phase in the first term of (11) by a corresponding configuration of vector potential $\mathbf{A}$, which, as shown by Abrikosov [24], makes this defect being of finite energy per unit length. The corresponding solution for the vector potential for a vortex characterized by 2π phase change around its core is following:

$$\mathbf{A} = \frac{\mathbf{r} \times \mathbf{e}_z}{|r|}|\mathbf{A}(r)| \quad (12)$$

$$|\mathbf{A}(r)| = \frac{1}{er} - \sqrt{\frac{\Psi^2}{m}}K_1\left(e\sqrt{\frac{\Psi^2}{m}}r\right) \quad (13)$$

where r measures distance from the core and $\mathbf{e}_z$ is a unit vector pointing along the core.

As it was discussed in [24] the above solution for $\mathbf{A}$ corresponds to the situation when a vortex carries a magnetic flux quantum $\Phi_0 = 2\pi/e$. Thus the flux quantization in the ordinary superconductor is an effect rooted in topological and energetical considerations.

Let us examine now the two-gap model (1) in case of $\eta = 0$ and show that the two-gap Abelian Higgs model can not generate magnetic flux which would compensate

gradients of both order parameters in case of a general topological defect.

Let us consider a vortex in (1) such that the phase ϕ_1 has 2π winding around core while ϕ_2 is constant. Then if a system is "trying" to compensate gradient of ϕ_1 in the first term by a corresponding configuration of $\mathbf{A}$ (that is, by generating flux) at the same time the field $\mathbf{A}$ is inducing divergence in the second term. So, by generating exactly one flux quantum the system may make the first term being finite but it would induce a divergence in the second term. On the other hand, by decreasing flux, the system may minimize divergence in the second term but then the gradient in the first term is not compensated by $\mathbf{A}$ identically and also becomes divergent. Thus there are no finite energy solutions for this type of vortices. However, still there is a solution with a minimal energy [17] (a reminder of derivation is given below), which indeed should be weakly divergent like in a neutral $U(1)$ system. Apparently also in the case where the phase ϕ_1 has 2π winding around core while ϕ_2 has a -2π winding, the system may minimize one of the gradient terms by generating flux but at the same time it would increase divergence in the second term. That is because to minimize the configuration with winding 2π one should have magnetic flux going in one direction while for the case -2π in opposite direction. In the case where the phase ϕ_1 has $2\pi n$ winding around core and ϕ_2 also has $2\pi n$ winding the system can generate flux which would compensate both gradients [17]. Only this type of topological defects in this system has finite energy.

Let us now describe formally the vortices with fractional flux following [17]. In the London limit ($|\Psi_\alpha|$=const) the eqs. (1), (2) become:

$$F = \frac{\rho^2}{4}(\nabla \vec{\mathbf{n}})^2 + \frac{\rho^2}{16}\vec{\mathsf{C}}^2 + \frac{1}{32e^2}[\nabla_i \mathsf{C}_j - \nabla_j \mathsf{C}_i]^2 + \rho^2 K n_1 =$$
$$\frac{\rho^2}{4}\sin^2 \tilde{\theta}(\nabla \gamma_n)^2 + \rho^2 \left[\sin^2\left(\frac{\tilde{\theta}}{2}\right)\nabla\phi_1 + \cos^2\left(\frac{\tilde{\theta}}{2}\right)\nabla\phi_2 - e\mathbf{A}\right]^2$$
$$+\frac{\mathbf{B}^2}{2} + \rho^2 K \sin\tilde{\theta}\cos(\phi_1 - \phi_2) \tag{14}$$

Eq. (14) can also be expressed in traditional variables:

$$F = \frac{1}{2}\frac{\frac{|\Psi_1|^2}{m_1}\frac{|\Psi_2|^2}{m_2}}{\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}}(\nabla(\phi_1 - \phi_2))^2 +$$
$$\frac{2}{\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}}\left\{\frac{|\Psi_1|^2}{2m_1}\nabla\phi_1 + \frac{|\Psi_2|^2}{2m_2}\nabla\phi_2 - e\left(\frac{|\Psi_1|^2}{m_1}\right.\right.$$
$$\left.\left.+\frac{|\Psi_2|^2}{m_2}\right)\mathbf{A}\right\}^2 + \frac{\mathbf{B}^2}{2} + 2\eta|\Psi_1\Psi_2|\cos(\phi_1 - \phi_2) \tag{15}$$

which is a rearrangement of the variables in (1) with no approiximations involved. The first term, in the case when $\eta = 0$, describes a *neutral* boson (a coupling to $\mathbf{A}$ is eliminated when we extract the difference of gauge

invariant phase gradients) associated with $\vec{\mathbf{n}}$ (with $O(3)$ symmetry of $\vec{\mathbf{n}}$, in the London limit being broken down to $O(2)$ by mass terms for n_3). The mass term for n_1 breaks the remaining $O(2)$ symmetry. The second term is the same as the second term in (14), together with third term $\mathbf{B}^2/2 = (32e^2)^{-1}[\nabla_i \mathsf{C}_j - \nabla_j \mathsf{C}_i]^2$ it describes the massive vector field $\vec{\mathsf{C}}$.

Let us observe that the models (1) and (2) have four characteristic length scales - coherence lengths of condensates, magnetic field penetration length λ and also there is a length scale associated with intrinsic Josephson effect (which is inverse mass for n_1). Below we discuss a system where both coherence lengths are of the same order of magnitude and where the characteristic length scale associated with inverse mass for n_1 is much larger than magnetic field penetration length λ in type-II limit and much larger than coherence lengths in type-I limit (which amounts to a negligibly small influence of the Josephson coupling at length scales much smaller than the inverse mass for n_1). The discussion of the phase diagram of the system in the regime of large η will be presented in another publication [25]. We stress that in a rigorous sence we should have $\eta = 0$ in order to speak about true BKT transitions, however when η is nonzero but sufficiently small we can speak about some finite-size crossovers. We also note that in principle it is possible to have physical systems where η is exactly zero - such situation appears e.g. in theoretical studies of liquid metallic hydrogen (which appears being close to being realized in high pressure experiments) which should allow superconductivity of protonic and electronic Cooper pairs [14]. Since electrons can not become protons the condensates are independently conserved and we have exact $U(1) \times U(1)$ symmetry. So in the zero-η case, which we consider in this paper, the system (1), (2) has a *neutral* $U(1)$ symmetry associated with the variable $\gamma_n = \phi_1 - \phi_2$.

Vortices in this system, characterized by the following phase change around a core $\Delta\phi_1 = 2\pi k_1; \Delta\phi_2 = 2\pi k_2$, *in what follows we denote for brevity by* (k_1, k_2). From (14) and (15) one can observe that for vortices characterized by $\Delta(\phi_1 + \phi_2) \equiv \oint_\sigma dl[\nabla(\phi_1 + \phi_2)] = 4\pi m; \Delta(\phi_1 - \phi_2) = 0$, (where we integrate over a closed curve σ around a vortex core) the first term in (14) and (15) is identically zero and such vortices is a analogue of m-flux quanta Abrikosov vortices in an ordinary superconductor characterized by $\frac{|\Psi|^2}{m} = \left(\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}\right)$. It should be observed that if *both* phases $\phi_{1,2}$ change by 2π around the core then a vortex carries *one* quantum of magnetic flux.

Let us now consider the case when $\Delta\phi_1 = 2\pi$ and $\Delta\phi_2 = 0$ [a vortex (1,0)]. First of all such a vortex features a neutral superflow characterized by a 2π gain in the variable γ_n. As follows from (14), (15), the vortex in this respect is equivalent to a vortex in a neutral superfluid with superfluid stiffness $\frac{1}{2}\rho^2 \sin^2 \tilde{\theta} =$

$\frac{\Psi_1^2}{m_1}\frac{\Psi_2^2}{m_2}\left[\frac{\Psi_1^2}{m_1}+\frac{\Psi_2^2}{m_2}\right]^{-1}$. Besides that, as follows from (14) and (15), a topological defect $(1,0)$ is necessarily accompanied by a nontrivial configuration of the charged field $\vec{C}$. That is, for such a topological defect the second term in (14) becomes $\rho^2\left[\sin^2\left(\frac{\tilde\theta}{2}\right)\nabla\phi_1-e\mathbf{A}\right]^2$. Thus, this term is nonvanishing for such a vortex configuration, which means that this vortex besides neutral vorticity also carries magnetic field. Here we stress that a minimal energy solution for a vortex with a given winding number is a solution where the second term in (14) is made finite by a corresponding configuration of $\mathbf{A}$. Lets us calculate the magnetic flux carried by such a vortex. The supercurrent around the core of this vortex is:

$$\mathbf{J}=2e\rho^2\left[\sin^2\left(\frac{\tilde\theta}{2}\right)\nabla\phi_1-e\mathbf{A}\right]. \qquad (16)$$

Lets us now integrate this expression over a closed curve σ situated at a distance larger than λ from the vortex core. Indeed at a distance much larger than the penetration length the supercurrent $\mathbf{J}$, or equivalently the massive field $\vec{C}$, vanishes. Thus we arrive at the following equation:

$$\Phi=\sin^2\left(\frac{\tilde\theta}{2}\right)\oint_\sigma\nabla\phi_1 d\mathbf{l}=\sin^2\left(\frac{\tilde\theta}{2}\right)\Phi_0 \qquad (17)$$

where $\Phi=\oint_\sigma\mathbf{A}d\mathbf{l}$ is the magnetic flux carried by our vortex and $\Phi_0=2\pi/e$ is the standard magnetic flux quantum. Obtaining equations of motions for $\vec{C}$ from (14) we can find a solution for asymptotic behavior of the field $\vec{C}$ for this vortex. At a distance larger than coherence length $\xi=max[\xi_1,\xi_2]$ from the core the field $\vec{C}$ behaves as:

$$|\vec{C}|=\sin^2\left(\frac{\tilde\theta}{2}\right)\frac{\Phi_0}{4\pi^2 e\rho^2\lambda^3}K_1\left(\frac{r}{\lambda}\right), \qquad (18)$$

where r is the distance from the core. So this vortex has the following structure:

- $r<\xi$ vortex core;
- $\xi<r<\lambda$ this region features neutral superflow associated with the gradients of the variable γ_n as well as charged supercurrent associated with the field $\vec{C}$.
- $r>\lambda$ this region features neutral superflow (which like in a vortex in any neutral system vanishes as $1/r$ away from the core), whereas at the distance $r>\lambda$ the magnetic field and the field $\vec{C}$ vanish exponentially according to (18).

We can also derive the energy per unit length of this vortex which is:

$$E=E_c+\left[\sin^2\left(\frac{\tilde\theta}{2}\right)\frac{\Phi_0}{4\pi\lambda}\right]^2\log\frac{\lambda}{\xi}+\frac{\pi}{2}(\sin\tilde\theta\ \rho)^2\log\frac{R}{\xi}(19)$$

where E_c is the core energy and R is the sample size. The second term in (19) is the energy of charged current and magnetic field which comes from the second and third terms in (15). The third term in (19) is the kinteic energy of neutral current which comes from the first term in (15). In an infinite sample, the energy per unit length (19), of this vortex in logarithmically divergent (like in a neutral superfluid) as a consequence of the existence of vorticity in the field of the massless neutral boson. When $\eta\neq 0$ vortex energy diverges linearly.

We would like to stress that since the two $U(1)$ symmetries are not independent in this system but are coupled by vector potential the system actually behaves like a system with a $U(1)$ neutral and a $U(1)$ gauged symmetries as it is seen from separation of variables (14), (15). The physical origin of the massless neutral $U(1)$ boson and flux fractionalisation in the two-gap system can be understood also from the following considerations: We can write expressions for individual supercurrents in the London limit: $\mathbf{j}_\alpha=\frac{ie}{m_\alpha}|\Psi_\alpha|^2\nabla\phi_\alpha-\frac{e^2}{4m_\alpha}|\Psi_\alpha|^2\mathbf{A}$, $(\alpha=1,2)$ from where it explicitly follows that *even if there are gradients of ϕ_1 while there are no gradients of ϕ_2 nonetheless coupling by $\mathbf{A}$ induces current $\mathbf{j}_2$ in the condensate Ψ_2*. For the vortex $(1,0)$ such current partially compensates magnetic flux induced by Ψ_1 and this compensation leads to existence of fractional flux and an effective neutral superflow in the system. The fractional flux follows from the condition that at a distance larger tham λ from the vortex core the resulting charged current $\mathbf{J}=\mathbf{j}_1+\mathbf{j}_2$ should vanish. The physical meaning of the neutral superflow is the circulation of two sorts of charged Cooper pairs in *opposite* directions where the charged individual currents $\mathbf{j}_\alpha$ compensate magnetic field induced by each other.

We also remark that a straightforward inspection of (2) shows that this model could allow neutral vortices associated with the neutral $O(2)$ boson without a nontrivial configuration of the field $\vec{C}$ for any values of $\tilde{n}_3$. However such solutions (e.g. vortices in the case $\tilde{n}_3\neq 0$ characterized by $\Delta\gamma_n=2\pi;\vec{C}\equiv 0$) are unphysical because these vortices do not satisfy the condition that ϕ_i changes by $2\pi k_i$ around the vortex core (as follows from (14)). Thus, while the original model (1) and the extended Faddeev model (2) have the same number of degrees of freedom so that the fields $\vec{n}$ and $\vec{C}$ are dynamically independent by construction, the mapping incurs a constraint on topological defects in $\vec{n}$ and $\vec{C}$: when we go around the vortex core the two phases ϕ_i should have $2\pi n_i$ gains (with n_i being necessary integer).

BEREZINSKII-KOSTERLITZ-THOULESS TRANSITIONS IN TWO-GAP SYSTEM

We begin with a remark that as it is well known, formally, the Abelian Higgs model $H = \frac{1}{2m}|(i\nabla - e\mathbf{A})\Psi)|^2 + a\Psi^2 + \frac{b}{2}\Psi^4$ does not exhibit the BKT transition due to interaction between Abrikosov vortices is of short range, being screened over the magnetic field penetration length λ [3] (see also a remark [26] and a review [1]). However when the penetration length is significantly large the system undergoes a "would be" BKT transition or crossover (see a review [1]) approximately at the temperature $T^{BKT} = \frac{\pi}{2}\frac{\Psi^2(T^{BKT})}{m}$. The equation for T^{BKT} should be solved self-consistently with equation for the temperature-dependent gap modulus $|\Psi(T)|$. The gap modulus $|\Psi(T)|$ opens at a characteristic temperature $T^* > T^{BKT}$, and albeit the phase is random at $T^* > T > T^{BKT}$, however $|\Psi(T)|$ has roughly the meaning of the measure of the density of preformed Cooper pairs in the system [27]. The "would be" BKT transition in a charged system is experimentally observable in extreme type-II superconductors [1, 28]. This transition/crossover is associated with onset of superconductivity in an extreme type-II planar system. In contrast a system with a short penetration length does not exhibit planar superconductivity at nonzero temperature.

Let us now discuss the phase diagram of planar $U(1) \times U(1)$ superconductor in the type-I limit where both coherence lengths are larger or equal to magnetic field penetration length:

Phase diagram of a planar two-gap system in the type-I limit

At finite temperature a planar system generates vortices and antivortices. The BKT transition manifests itself as formation of bound vortex-antivortex pairs. In the regime $\lambda \leq \xi$ we can neglect the presence of the second and third terms in (15) while describing interaction of vortices. The second and third terms describe charged supercurrent around the vortex core, it can be seen from (14) where second and third terms correspond to the second and third terms in (15) and thus these terms describe a massive vector field $\vec{\mathsf{C}}$. The inverse mass of this field is λ thus a solution of equations of motion for $\vec{\mathsf{C}}$ should be localized at the length scale λ and in the limit of small λ can be neglected (see also a discussion after eq. (18)). So, when the penetration length is very short then we can neglect interaction of vortices mediated by the charged current which is exponentially suppressed. On the other hand the neutral mode [described by the first term in (14), (15)] is not affected by smallness of penetration length. Thus effectively our system is described by the gradient term of the composite neutral mode which

is (we remind that we consider the case of the negligible small η):

$$F^{eff\ (type-I)} =$$
$$\frac{1}{2}\frac{|\Psi_1|^2}{m_1}\frac{|\Psi_2|^2}{m_2}\left[\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}\right]^{-1}(\nabla(\phi_1 - \phi_2))^2 \quad (20)$$

We observe that the system has the following vortices with *identical* configuration of the neutral superflow $(1,0)$ and $(0,-1)$ and antivortices $(-1,0)$ and $(0,1)$. A neutral superflow gives rise to a long-range logarithmic interaction between vortices and antivortices. So, these vortices undergo a genuine BKT transition at the temperature which can be derived immediately from the first term in (14) and (15):

$$T_{sf}^{BKT} = \frac{\pi}{2}\rho^2\sin^2\tilde{\theta} = \frac{\pi}{2}\frac{\frac{|\Psi_1|^2(T_{sf}^{BKT})}{m_1}\frac{|\Psi^2|_2(T_{sf}^{BKT})}{m_2}}{\frac{|\Psi_1|^2(T_{sf}^{BKT})}{m_1} + \frac{|\Psi_2|^2(T_{sf}^{BKT})}{m_2}} \quad (21)$$

which, in the case of two-gap superconductor, should be solved self-consistently with the equations for the gap modules $|\Psi_{1,2}(T)|$ [27]. We stress that in a rigorous sence we should have $\eta = 0$ in order to speak about BKT transitions, when η is nonzero then fractional vortices of the type (1,0) interact linearly at length scales larger than inverse mass for n_1 and thus do not exhibit true BKT transition. However when η is sufficiently small we can speak about some finite-size effects which, under certain conditions may even be observable in an experiment.

We should first emphasis that the transition (21) in type-I limit is associated with formation of vortex-antivortex pairs of the following types [(1,0) + (-1,0)], [(1,0) + (0,1)], [(-1,0) + (0,-1)], [(0,-1) + (0,1)] (due to e.g. the vortices (1,0) and (0,-1) have equivalent configuration of composite neutral superflow in the limit $\lambda \rightarrow \xi$, and thus in Coulomb gas mapping represent identical charge). We stress that the interaction of vortices with windings of phases belonging to different condensates like e.g. pairs [(1,0) + (0,1)] is mediated by neutral superflow which has a *composite* nature. That is, this superflow consists of particle current of both condensates (see a remark in conclusion of the previous section). Besides that the system allows additional relevant composite topological excitation: the one-flux-quantum vortices (aslo described in [17]) (1,1) and (-1,-1). These composite vortices are topologically equivalent to the pairs [(1,0) + (0,1)], [(-1,0) + (0,-1)]. The composite vortices (1,1) and (-1,-1) play an important role in this system: these vortices do not have neutral superflow (first term in (15) is identically zero for such a vortex configuration) thus the interaction of these vortices is exponentially screened, they have finite energy, and remain liberated at any nonzero temperature. Thus, due to existence of the mentioned above four equivalent vortex pairing channels and the relevant composite vortices (1,1)

and (-1,-1), the phase transition (21) marks onset of the quasi-long-range order only in the variable $\gamma_n = \phi_1 - \phi_2$ (described by first term in (14), (15)), while the variable $\gamma_c = \sin^2(\frac{\tilde{\theta}}{2})\phi_1 + \cos^2(\frac{\tilde{\theta}}{2})\phi_2$ associated with charged boson in (14),(15) remains disordered. In the system considered by Rodriguez [10] the counterpart of this transition is a transition in the physical electronic phase (the superconducting transition) thus, the initially massless composite mode in [10] is actually coupled to gauge field like in an ordinary one gap superconductor and the phase transition in the composite field is transformed in a "would be" BKT superconductive transition/crossover like in a one-gap Abelian Higgs model. In contrast, we stress the two-gap superconductor is a physical system which possesses a genuine and potentially experimentally observable massless neutral boson associated with the composite phase field γ_n. Let us once more stress the meaning of the field γ_n:

- In a two-gap system variables can be separated into a phase difference and a phase sum. Only the phase sum is coupled to vector potential.

- The gradients of phase sum are gauged away at the length scale λ. (Just like in ordinary Abrikosov vortex a phase gradient is compensated by a vector potential at the penetration length scale).

- When λ is small the long- range interaction is mediated only by gradients of phase difference which is not coupled to vector potential and thus is of long range and determines the energy of e.g. an $(1,0)+(0,1)$ pair. When such pairing occurs the individual phases are indeed disordered [in contrast to the case if pairing would be only of type $(1,0)+(-1,0)$ and $(0,1)+(0,-1)$] and quasi-long-range order sets in only in phase difference. Away from a pair $(1,0)+(0,1)$ the phase difference is not characterized by a nontrivial winding number and its gradients do not contribute to kinetic terms of the GL functional. At the same time the phase sum has a nontrivial winding but it is compensated by vector potential thus it gives only a finite contribution to the GL free energy which is not important for the BKT transition (simliar like a finite energy contribution from vortex cores). So the system can be mapped onto a Coulomb gas in a standard way.

We should stress that in two-gap superconductor (1) the transition in the field γ_n has little to do with superconductivity since it is a transition in the composite neutral field. It marks onset of what should be rather called superfluidity in the field of the composite neutral boson. Also the important fact is that an elementary topological defect with neutral vorticity in the models (14), (15) also carries a fraction of magnetic flux quantum. Thus albeit we have the phase ordering transition in the neutral field however the fractional flux which accompanies the vortices [17] allows a detection of this *superfluid* phase transition in a two-gap superconductor by standard experimental technique used in studies of *charged* systems. That is, the vortex-antivortex interaction is mediated only by composite neutral superflow while the magnetic flux (at length scale λ from the core) allows observation of these defects in flux-noise measurements or measurements in applied dynamic magnetic field. Besides that because of the existence of neutral mode, the transition (21) in e.g. flux-noise measurements should manifest itself as a sharp well-defined transition which can not be observed in ordinary charged systems where one can see only a "would be" BKT transition, "washed out" by Meissner effect. So, a two-gap superconductor displays unique potentially experimentally observable novel phase: in a type-I limit two-gap superconductor should exhibit quasi-superfluidity at the same time it should not exhibit true superconductivity at any nonzero temperature. In the type-I limit the true superconductivity sets in at $T \to 0$ because at zero temperature the dimensional reduction does not work and the system is effectively 3-dimensional. In the intermediate case when penetration length is not negligibly small there could occur a washed out crossover to a superconducting phase at a nonzero characteristic temperature T_c $(T_{sf} >> T_c > 0)$.

Let us now discuss the phase diagram of a two-gap system in the case of a large penetration length.

Phase diagram of a planar type-II two-gap system.

When penetration length is much larger than coherence lengths, the system, again, possesses the following vortex excitations: (1,0), (-1,0), (0,1), (0,-1) which still all interact with each other by means of the composite neutral superflow. Also there exist composite vortices without neutral superflow (1,1) and (-1,-1). In contrast to the type-I limit, at larger λ the system has *two* energetically preferred vortex-antivortex pairing channels (instead of four equivalent vortex pairing channels) associated with binding of the following pairs $[(1,0) + (-1,0)]$ and $[(0,-1) + (0,1)]$. It is because, as it follows from (14) and (15) the vortices (1,0) and (-1,0) have attractive interaction *both* due to neutral superflow and charged current, in contrast the vortices (1,0) and (0,-1) interact only due to their neutral modes [which is a composite mode, whereas the charged supercurrent which accompanies the vortices (1,0) and (0,-1) consists only of Cooper pairs of sort "1" in the case (1,0) and sort "2" in the case (0,-1) (see the concluding remark in section III)].

Let us illustrate the discussion with the vortex (1,0). The free energy density for this configuration becomes:

$$F = \frac{1}{2}\frac{|\Psi_1|^2}{m_1}\frac{|\Psi_2|^2}{m_2}\left[\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2}\right]^{-1}(\nabla\phi_1)^2 +$$

$$2 \left[\frac{|\Psi_1|^2}{m_1} + \frac{|\Psi_2|^2}{m_2} \right]^{-1} \left\{ \frac{|\Psi_1|^2}{2m_1} \nabla\phi_1 - 2e\mathbf{A} \right\}^2$$

$$+ \frac{\mathbf{B}^2}{2}. \tag{22}$$

From this expression it is seen that the interaction of vortices (1,0) and (-1,0) at the length scales smaller than λ is characterized by the effective stiffness

$$J_1^{eff} = J^n + J_1^c, \quad \text{where } J^n(T) = \frac{\frac{|\Psi_1|^2(T)}{m_1} \frac{|\Psi_2|^2(T)}{m_2}}{\frac{|\Psi_1^2|(T)}{m_1} + \frac{|\Psi_2^2|(T)}{m_2}} \tag{23}$$

and $J_1^c(T) = \dfrac{\frac{|\Psi_1|^4(T)}{m_1^2}}{\frac{|\Psi_1|^2(T)}{m_1} + \frac{|\Psi_2|^2(T)}{m_2}} - \begin{pmatrix} corrections \ due \\ to \ finite \ \lambda \end{pmatrix}$ (24)

The contribution to the interaction between these vortices which is mediated by charged current (characterized by the stiffness J^c) should receive corrections due to charged current vanishes over the magnetic field penetration length λ. Similarly, the effective stiffness for interaction between vortices (0,1) and (0,-1) is given by an analogous expression

$$J_2^{eff} = J^n + J_2^c. \tag{25}$$

where

$$J_2^c(T) = \frac{\frac{\Psi_2^4(T)}{m_2^2}}{\frac{\Psi_1^2(T)}{m_1} + \frac{\Psi_2^2(T)}{m_2}} - \begin{pmatrix} corrections \ due \\ to \ finite \ \lambda \end{pmatrix} \tag{26}$$

Thus the system in the regime $\lambda \to \infty$ will have two transitions at the following temperatures:

$$T_1^{BKT} = \frac{\pi}{2} J_1^{eff}(T_1^{BKT})$$
$$T_2^{BKT} = \frac{\pi}{2} J_2^{eff}(T_2^{BKT}) \tag{27}$$

This is a consequence of the fact that making λ large, reduces four vortex pairing channels (which are equivalent in the type-I limit) to two non-equivalent energetically preferred pairing channels. We should emphasis that the composite co-centered excitations (1,1) and (-1,-1) which do not carry superflow [and which are partially responsible for disordering the variable $\gamma_c = \sin^2(\frac{\tilde{\theta}}{2})\phi_1 + \cos^2(\frac{\tilde{\theta}}{2})\phi_2$ in the short penetration length limit] are irrelevant if the penetration length is large since the effective stiffness of interaction between vortices (1,1) and (-1,-1) is $J = [|\Psi_1|^2/m_1 + |\Psi_2|^2/m_2]$ (as it follows from (15), (14)) and thus when we can not assume that interaction is completely screened due to Meissner effect, then the large effective stiffness which characterizes interaction of these defects prevents existence of liberated vortices of this type at the temperatures (27). We also remark that in the type-II limit the interaction of vortex pairs of the type $(1,0) + (0,1)$ is getting depleted at the length scales $r < \lambda$ as shown in appendix A.

CONCLUSION

We presented a discussion of the phase diagrams of planar $U(1) \times U(1)$ superconductors and predicted what could be a novel quasi-superfluid state realized in two-gap superconductor in a type-I limit. The physical meaning of the quasi-superfluid state in a two-component charged system is a quasi-long range order in phase difference when individually the phases are disordered. Such a state leads to a counter-intuitive situation of purely topological origin when there are no disspationless individual currents while a superflow of oppositely directed two charged currents is dissipationless.

We also stress that the present discussion is relevant for much wider class of physical models than two-gap superconductivity: the models (1), (2) are also relevant in the high energy physics [19, 21].

A note added in proof: We call the reader's attention to a problem of a BKT transition similar to the discussed above type-II limit, which appears in high-T_c superconductors and was considered in [29] and references cited therein. We thank G. Blatter for informing us about these references. Another, related from a field theoretic point of view, problem was considered in [30].

APPENDIX A

The pairing of vortices and antivortices of the type $(1,0) + (0,1)$ is possible because two condensates are coupled by vector potential so gauge invariant equations for currents of two types of Cooper pairs are: $\mathbf{j}_\alpha = \frac{ie}{m_\alpha}|\Psi_\alpha|^2\nabla\phi_\alpha - \frac{e^2}{4m_\alpha}|\Psi_\alpha|^2\mathbf{A}$, ($\alpha = 1,2$) so that if there are gradients of ϕ_1 while there are no gradients of ϕ_2 nonetheless coupling by $\mathbf{A}$ induces current $\mathbf{j}_2$ in the condensate Ψ_2. For this reason a vortex (1,0) interacts with a a vortex with phase winding in different condensate: $(0,1)$. Let us now consider a vortex $(1,0)$. It has

the following particle currents of condensates $\Psi_{1,2}$ around its core:

$$\mathbf{j}_1 = \frac{ie}{m_1}|\Psi_1|^2\nabla\phi_1 - \frac{e^2}{4m_\alpha}|\Psi_1|^2\mathbf{A},$$

$$\text{and} \qquad \mathbf{j}_2 = -\frac{e^2}{4m_\alpha}|\Psi_2|^2\mathbf{A} \qquad (28)$$

The standard solution for the vector potential of an Abrikosov vortex is:

$$\mathbf{A} = \frac{\mathbf{r}\times\mathbf{e}_z}{|r|}|\mathbf{A}(r)| \qquad (29)$$

where r measures distance from the core and $\mathbf{e}_z$ is a unit vector pointing along the core. For such a vortex the solution for vector potential is

$$|\mathbf{A}| = \frac{\frac{\Psi_1^2}{m_1}}{\frac{\Psi_1^2}{m_1}+\frac{\Psi_2^2}{m_2}}\frac{1}{er} - \frac{\frac{\Psi_1^2}{m_1}}{\sqrt{\frac{\Psi_1^2}{m_1}+\frac{\Psi_2^2}{m_2}}}K_1\left(e\sqrt{\frac{\Psi_1^2}{m_1}+\frac{\Psi_2^2}{m_2}}r\right) \quad (30)$$

For $r << \lambda = \left(e\sqrt{\frac{\Psi_1^2}{m_1}+\frac{\Psi_2^2}{m_2}}\right)^{-1}$ eq (30) reads

$$|\mathbf{A}| \approx \frac{\frac{\Psi_1^2}{m_1}}{\sqrt{\frac{\Psi_1^2}{m_1}+\frac{\Psi_2^2}{m_2}}}\left[\frac{1}{2}\ln\left(\frac{r}{2\lambda}\right) - \frac{1-2\gamma}{4}\right]\frac{r}{\lambda} \qquad (31)$$

For this reason for the mentioned above vortex pair, in a type-II superconductor with the separation r when $\lambda >> r >> \max_i[\xi_i(T_{\text{BKT}})]$, the interaction mediated by the neutral mode due to coupling by vector potential, which is composed of oppositely circulating supercurrents of two species of Cooper pairs, is depleted because a vortex $(1,0)$ at length scale $r << \lambda$ has depleted current $\mathbf{j}_2$ likewise a vortex $(0,1)$ also has depleted current $\mathbf{j}_1$.

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